Charm Mass Dependence of the $\mathcal{O}(\alpha_s^2 n_f)$ Correction to Inclusive $B \to X_c e \overline{\nu}_e$ Decay

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Abstract

We compute the $\alpha_s^2 n_f$ perturbative QCD contribution to semileptonic B decay, including the finite mass of the charm quark. This result provides an estimate of the size of the two-loop correction, which is found to be about 50% of the one-loop correction. We use these results to set the scale for the one-loop correction using the scheme of Brodsky, Lepage and Mackenzie and find a BLM scale of $\mu_{\rm BLM} = 0.13\, m_b$, when the inclusive semileptonic rate is expressed in terms of the b and c quark pole masses and the $\overline{\rm MS}$ strong coupling. The two loop correction lies roughly midway between that obtained at $m_c = 0$ and that obtained in the Shifman-Voloshin limit $m_b, m_c \gg m_b - m_c \gg \Lambda_{\rm QCD}$ while the corresponding BLM scale is somewhat closer to that obtained in the former case.

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1. Introduction

Inclusive semileptonic B decays provide a method for determining the magnitude of the element of the Cabibbo–Kobayashi–Maskawa matrix V_{cb}. In the limit where the b quark mass is much larger than the QCD scale the B meson decay rate is equal to the b quark decay rate [1]. Corrections to this first arise at order $(\Lambda_{QCD}/m_b)^2$ and these nonperturbative corrections may be written in terms of the matrix elements [2]–[4] $\langle B|\bar{b}(iD)^2 b|B\rangle$ and $\langle B|\bar{b}ig\sigma_{\mu\nu}G^{\mu\nu}b|B\rangle$. The order α_s corrections to the decay rate, including the complete c quark mass dependence, have been calculated in [5].

No complete calculation of the two-loop perturbative corrections to inclusive heavy quark decay has been performed. However, the $\mathcal{O}(\alpha_s^2 n_f)$ correction to the inclusive semileptonic decay rate to a massless quark has been recently calculated [6]. This correction is of interest in its own right for two reasons. First, if one adopts the viewpoint of Brodsky, Lepage and Mackenzie [7], one can use this result to set the appropriate scale for α_s in the leading term. For $b \to X_u$ semileptonic decay, this leads to the surprisingly low scale $\mu_{\rm BLM} = 0.07\, m_b$ [6]. Second, even if one does not adopt this viewpoint, we argue that this calculation provides a good estimate of the size of the two-loop calculation. This is not because n_f is large, but rather because the QCD beta function $\beta = 11 - 2n_f/3$ is large, so the vacuum polarization graphs which contribute to this term are be expected to dominate the two-loop result. Empirically, this is certainly true for $R(e^+e^- \to \text{hadrons})$, $\Gamma(\tau \to \nu_\tau + \text{hadrons})$ and the two-loop relation between the pole mass and the $\overline{\rm MS}$ mass of a heavy quark:

$$
\begin{aligned}
&R(e^+e^- \to \text{hadrons}) = 3\left(\sum_i Q_i^2\right)\left[1 + \frac{\overline{\alpha}_s(\sqrt{s})}{\pi} + (0.17\beta + 0.08)\left(\frac{\overline{\alpha}_s(\sqrt{s})}{\pi}\right)^2 + ...\right] \\
&\frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{3\Gamma(\tau \to \nu_\tau \overline{\nu}_e e^-)} = 1 + \frac{\overline{\alpha}_s(m_\tau)}{\pi} + (0.57\beta + 0.08)\left(\frac{\overline{\alpha}_s(m_\tau)}{\pi}\right)^2 + ... \\
&\frac{m_Q^{\rm pole}}{m_Q^{\overline{MS}}(m_Q)} = 1 + \frac{4}{3}\frac{\overline{\alpha}_s(m_Q)}{\pi} + (1.56\beta - 1.05)\left(\frac{\overline{\alpha}_s(m_Q)}{\pi}\right)^2 + ...
\end{aligned}
\tag{1.1}
$$

where $\overline{\alpha}_s$ is the $\overline{\rm MS}$ strong coupling. In each of these cases, the $\mathcal{O}(\alpha_s^2\beta)$ term provides an excellent approximation to the full two-loop correction. Using the term proportional to β as an estimate of the size of the two-loop QCD corrections for $b \to X_u$ semileptonic decay

and taking $n_f = 3$[†] gives [6]

$$\begin{aligned}\Gamma(b \to X_u e\overline{\nu_e}) &= |V_{bu}|^2 \frac{G_F^2 m_b^5}{192\pi^3}\left[1 - \left(\frac{\overline{\alpha}_s(m_b)}{\pi}\right)[2.41] - \left(\frac{\overline{\alpha}_s(m_b)}{\pi}\right)^2 [28.7] + ...\right] \\ &= |V_{bu}|^2 \frac{G_F^2 m_b^5}{192\pi^3}[1 - 0.15 - 0.11 + ...]\end{aligned} \tag{1.2}$$

where we have used $\overline{\alpha}_s(m_b) \simeq 0.2$. The two-loop term is clearly significant.

For charmed hadrons the situation is even worse. Using $\overline{\alpha}_s(m_c) = 0.29$, we find

$$\Gamma(c \to X_d e\overline{\nu_e}) = |V_{cd}|^2 \frac{G_F^2 m_c^5}{192\pi^3}[1 - 0.22 - 0.25 + ...] \tag{1.3}$$

and it does not appear that the perturbation series is controlled at all. This is reflected in the small size of the corresponding BLM scale $\mu_{\rm BLM} \sim 100\,{\rm MeV}$.

The results of [6] raise questions about the applicability of perturbative QCD to describe the semileptonic decay of charmed hadrons, and also suggest that two-loop corrections are important for a reliable extraction of V_{bc} from inclusive $b \to X_c$ decays. In this paper we extend these results to include the effects of the finite c quark mass in $b \to X_c$ decays. We find that, for $m_c/m_b = 0.3$, the BLM scale is raised to $\mu_{\rm BLM} = 0.13\,m_b$, and the corresponding perturbation series is

$$\begin{aligned}\Gamma(b \to X_c e\overline{\nu_e}) &= |V_{bc}|^2 \frac{G_F^2 m_b^5}{192\pi^3}[0.52]\left[1 - \left(\frac{\overline{\alpha}_s(m_b)}{\pi}\right)[1.67] - \left(\frac{\overline{\alpha}_s(m_b)}{\pi}\right)^2 [15.1] + ...\right] \\ &= |V_{bc}|^2 \frac{G_F^2 m_b^5}{192\pi^3}[1 - 0.11 - 0.06 + ...]\end{aligned} \tag{1.4}$$

The finite charm quark mass thus reduces the size of the higher-order corrections, although they are still significant.

In Eqs. (1.2)–(1.4), m_b and m_c are the pole masses of the b and c quark. If $\overline{\rm MS}$ masses are used in (1.2) and (1.3) the convergence is slightly improved [6].

† We have not taken into account the c quark mass for virtual c quarks. The true result will be somewhere between $n_f = 3$ and $n_f = 4$, and we have chosen to take the number of light flavours to be three for both c and b decays.

2. Calculation

The semileptonic decay rate of a b-quark is given by

$$\Gamma(b\to X_c e\overline{\nu_e}) = |V_{bc}|^2\,\frac{G_F^2 m_b^5}{192\pi^3} f\left(\frac{m_c}{m_b}\right)\left[1-\left(\frac{\overline{\alpha}_s(m_b)}{\pi}\right)\frac{2}{3}\left(\pi^2-\frac{25}{4}+\delta_1\left(\frac{m_c}{m_b}\right)\right)\right.$$
$$\left.-\left(\frac{\overline{\alpha}_s(m_b)}{\pi}\right)^2\left(\beta\,\chi_\beta\left(\frac{m_c}{m_b}\right)+\chi_0\left(\frac{m_c}{m_b}\right)\right)+\ ...\right] \tag{2.1}$$

where $\beta = 11-\frac{2}{3}n_f$, $\overline{\alpha_s}$ is the $\overline{MS}$ strong coupling, $f(x) = (1-x^4)(1-8x^2+x^4)-24x^4\log(x)$ and $m_{b,c}$ are the heavy quark pole masses. The function $\delta_1(x)$ has been computed in [5], and $\delta_1(0.3) = -1.11$. In this work we compute the function $\chi_\beta(x)$.

The second order corrections to the semileptonic decay rate have been shown [8] to be calculable from the first order corrections when a finite gluon mass is included. If the first order correction with a finite gluon mass m_g is denoted by $\Gamma^{(1)}(m_g)$ then the second order correction proportional to the perturbative QCD beta function is

$$\Gamma_\beta^{(2)} = -\beta\frac{\alpha_s^{(V)}(m_Q)}{4\pi}\int_0^\infty \frac{d\mu^2}{\mu^2}\left(\Gamma^{(1)}(\mu)-\frac{m_Q^2}{\mu^2+m_Q^2}\Gamma^{(1)}(0)\right)\quad , \tag{2.2}$$

where $\alpha_s^{(V)}$ is the strong coupling constant evaluated in the “V-scheme” of [7] and is related to the $\overline{\mathrm{MS}}$ coupling constant $\overline{\alpha}_s$ by

$$\alpha_s^{(V)}(\mu) = \overline{\alpha}_s(\mu)+\frac{5}{3}\frac{\overline{\alpha}_s(\mu)^2}{4\pi}\beta+\ ...\ . \tag{2.3}$$

Both virtual and bremstrahlung graphs contribute to the first order correction. The virtual graphs were computed analytically, with the final integrations of the lepton q^2 and m_{gluon} performed numerically, while the phase space integrals for the bremstrahlung graphs were performed numerically.

The leading two-loop correction $\chi_\beta(m_c/m_b)$ is plotted in fig. 1. According to the BLM scheme [7], this term is absorbed into the first order correction by redefining the scale at which the first order result is evaluated to $\mu = \mu_{\text{BLM}}$, where

$$\mu_{\text{BLM}} = m_b\exp\left[-\frac{3\chi_\beta(m_c/m_b)}{\pi^2-\frac{25}{4}+\delta_1(m_c/m_b)}\right]. \tag{2.4}$$

The BLM scale is plotted as a function of m_c/m_b in fig. 2. At $m_c = 0$ we reproduce the result [6] $\mu_{\text{BLM}}(0) = 0.07\,m_b$, while for $m_c/m_b = 0.3$ we find $\mu_{\text{BLM}} = 0.13\,m_b$. This is

somewhat higher than the scale found for the case of massless final state quarks but is still considerably smaller than m_b.

A non-trivial check on our numerical results is obtained in the Shifman-Voloshin limit $m_b, m_c \gg m_b - m_c \gg \Lambda_{\rm QCD}$ [9]. In this limit the final hadronic states are dominated by the D and D^* mesons (which are degenerate to leading order in $1/m_c$). They are produced almost at rest, and the rates for these *exclusive* processes can be calculated in the heavy quark effective theory with no free parameters. This gives

$$\Gamma(B \to D, D^* e\overline{\nu_e}) = \frac{G_F^2 (m_b - m_c)^5}{60\pi^3} \left[|\eta_V|^2 + 3|\eta_A|^2 \right] + \mathcal{O}\left(\frac{(m_b - m_c)^6}{m_b} \right) \tag{2.5}$$

where the coefficients η_A and η_V arise in matching from the QCD currents to the heavy quark currents

$$\begin{aligned} \overline{c}\gamma^\mu b &\to \eta_V \overline{h}_c \gamma^\mu h_b \ + \ ... \\ \overline{c}\gamma^\mu \gamma_5 b &\to \eta_A \overline{h}_c \gamma^\mu \gamma_5 h_b \ + \ ... \ , \end{aligned} \tag{2.6}$$

and in this limit the heavy c and b quarks have the same 4-velocity. The matching coefficients have the perturbative expansion

$$\begin{aligned} \eta_V &= 1 + \frac{1}{3}\frac{\overline{\alpha}_s(m_b)}{\pi}\phi(m_c/m_b) + \left(\frac{\overline{\alpha}_s(m_b)}{\pi}\right)^2 \left[\frac{1}{72}\phi(m_c/m_b)\beta + ...\right] + ... \\ \eta_A &= 1 + \frac{1}{3}\frac{\overline{\alpha}_s(m_b)}{\pi}\left[\phi(m_c/m_b) - 2\right] + \left(\frac{\overline{\alpha}_s(m_b)}{\pi}\right)^2 \left[\left(\frac{5}{72}\phi(m_c/m_b) - \frac{14}{72}\right)\beta + ...\right] + ... \end{aligned} \tag{2.7}$$

where the leading term was calculated in Refs. [9] and [10] and the $\mathcal{O}(\alpha_s^2\beta)$ terms were calculated in Ref. [11]. The function $\phi(z)$ is defined by

$$\phi(z) = -3\frac{1+z}{1-z}\log z - 6 \tag{2.8}$$

and so $\phi(1) = 0$. Furthermore, at $m_c = m_b$ the vector current is not renormalized, and Eq. (2.7) simplies to

$$\begin{aligned} \eta_V &= 1 \\ \eta_A &= 1 - \frac{2}{3}\frac{\overline{\alpha}_s(m_b)}{\pi} - \frac{7}{36}\left(\frac{\overline{\alpha}_s(m_b)}{\pi}\right)^2 \left[\beta + ...\right] + ... \end{aligned} \tag{2.9}$$

which gives

$$\begin{aligned} \Gamma(B \to D, D^* e\overline{\nu}_e) =& \frac{G_F^2 (m_b - m_c)^5}{15\pi^3} \left(1 - \frac{\overline{\alpha}_s(m_b)}{\pi} - \frac{7}{24}\left(\frac{\overline{\alpha}_s(m_b)}{\pi}\right)^2 \left[\beta + ...\right] + ...\right) \\ &+ \mathcal{O}\left[\frac{(m_b - m_c)^6}{m_b}\right]. \end{aligned} \tag{2.10}$$

The one loop term is in agreement with the results of Jezabek and Kuhn [5], while the two-loop term is in agreement with the limiting value of our inclusive calculation. This gives a BLM scale in the SV limit of $\mu_{\rm BLM} = 0.56\, m_b$.

3. Discussion and Conclusions

The BLM scale is one measure of the size of higher order corrections. However, as we argued in the introduction, we can also use our results as an estimate of the size of the two-loop corrections with $\mu = m_b$. For $m_c/m_b = 0.3$, the inclusive rate at leading order in $1/m_b$ is

$$\begin{aligned}
\Gamma(b \to X_c e \overline{\nu_e}) &= |V_{bc}|^2 \frac{G_F^2 m_b^5}{192\pi^3} [0.52] \left[1 - \left(\frac{\overline{\alpha}_s(m_b)}{\pi} \right) [1.67] - \left(\frac{\overline{\alpha}_s(m_b)}{\pi} \right)^2 [15.1] \ + \ ... \right] \\
&= |V_{bc}|^2 \frac{G_F^2 m_b^5}{192\pi^3} [0.52] \left[1 - 0.11 - 0.06 + ... \right] \\
&= |V_{bc}|^2 \frac{G_F^2 m_b^5}{192\pi^3} [0.52] \left[0.83 + ... \right]
\end{aligned} \tag{3.1}$$

where the factor of 0.52 arises from the lowest order phase space factor $f(0.3)$ and again we have taken $\overline{\alpha}_s(m_b) = 0.20$. The one and two loop corrections relative to the leading term are plotted as functions of m_c/m_b in fig. 3. We can compare this with our previous results for $b \to X_u$ semileptonic decay, Eq. (1.2). The finite charm quark mass somewhat reduces the size of the two-loop term, but it is still significant, about 50% of the $\mathcal{O}(\alpha_s)$ correction. The two loop correction lies roughly midway between that obtained at $m_c = 0$ and that obtained in the Shifman-Voloshin limit while the corresponding BLM scale is somewhat closer to that obtained in the former case.

We have expressed the $b \to X_c$ decay rate in terms of the b and c quark pole masses. This is convenient since the difference in these masses is determined by the meson masses (up to corrections of $\mathcal{O}(1/m_{c,b})$):

$$m_b - m_c = m_B - m_D + \mathcal{O}\left(1/m_{b,c}\right) \tag{3.2}$$

and the function $f(m_c/m_b)$ is less sensitive to the sum of the quark masses than the difference.

For charm decays, including the finite mass of the s quark has a negligible effect on the magnitude of the two-loop corrections. Because the s quark is so light its pole mass

is not a useful quantity. However, for illustrative purposes we have plotted the size of the one and two loop corrections relative to the leading term as functions of m_s/m_c in fig. 4, using $\overline{\alpha}_s(m_c) = 0.29$. Note that for small m_s, the two loop correction is larger than the one loop correction. For $m_s/m_c = 0.13$ the inclusive semileptonic decay rate is

$$\begin{aligned}\Gamma(c \to X_s e \overline{\nu_e}) &= |V_{cs}|^2 \frac{G_F^2 m_c^5}{192\pi^3} [0.87] \left[1 - \left(\frac{\alpha_s}{\pi}\right) [2.08] - \left(\frac{\alpha_s}{\pi}\right)^2 [22.7] + ... \right] \\ &= |V_{cs}|^2 \frac{G_F^2 m_c^5}{192\pi^3} [0.87] \left[1 - 0.19 - 0.19 + ... \right] .\end{aligned} \tag{3.3}$$

The perturbation series is still clearly uncontrolled.

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Figure Captions

Fig. 1. The function $\chi_\beta(m_c/m_b)$ from Eq. (2.1) plotted vs. m_c/m_b.

Fig. 2. The BLM scale as a function of m_c/m_b.

Fig. 3. The $\mathcal{O}(\alpha_s)$ (dashed line) and $\mathcal{O}(\alpha_s^2\beta)$ (solid line) corrections to the expansion of $\Gamma(b \to X_c e \overline{\nu}_e)$ relative to the leading term, plotted as functions of m_c/m_b. We are using $\overline{\alpha}_s(m_b) = 0.20$ and $n_f = 3$.

Fig. 4. The $\mathcal{O}(\alpha_s)$ (dashed line) and $\mathcal{O}(\alpha_s^2\beta)$ (solid line) corrections to the expansion of $\Gamma(c \to X_s e \overline{\nu}_e)$ relative to the leading term, plotted as functions of m_s/m_c. We are using $\overline{\alpha}_s(m_c) = 0.29$ and $n_f = 3$.

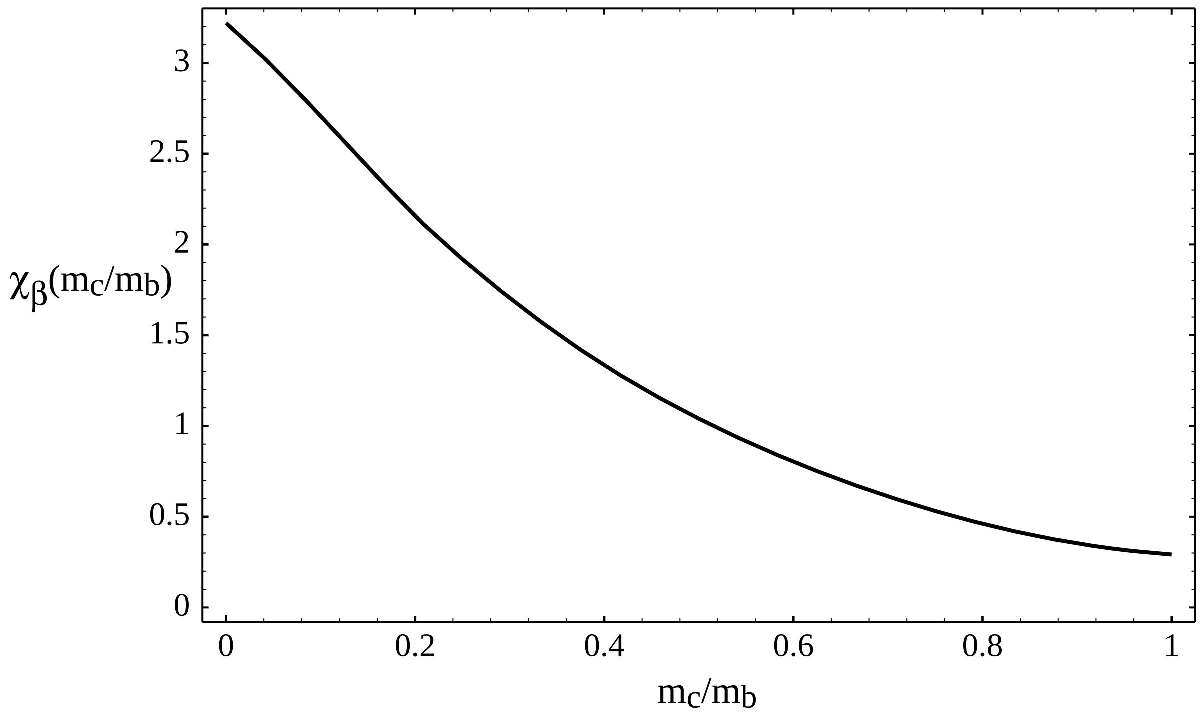


Figure 1

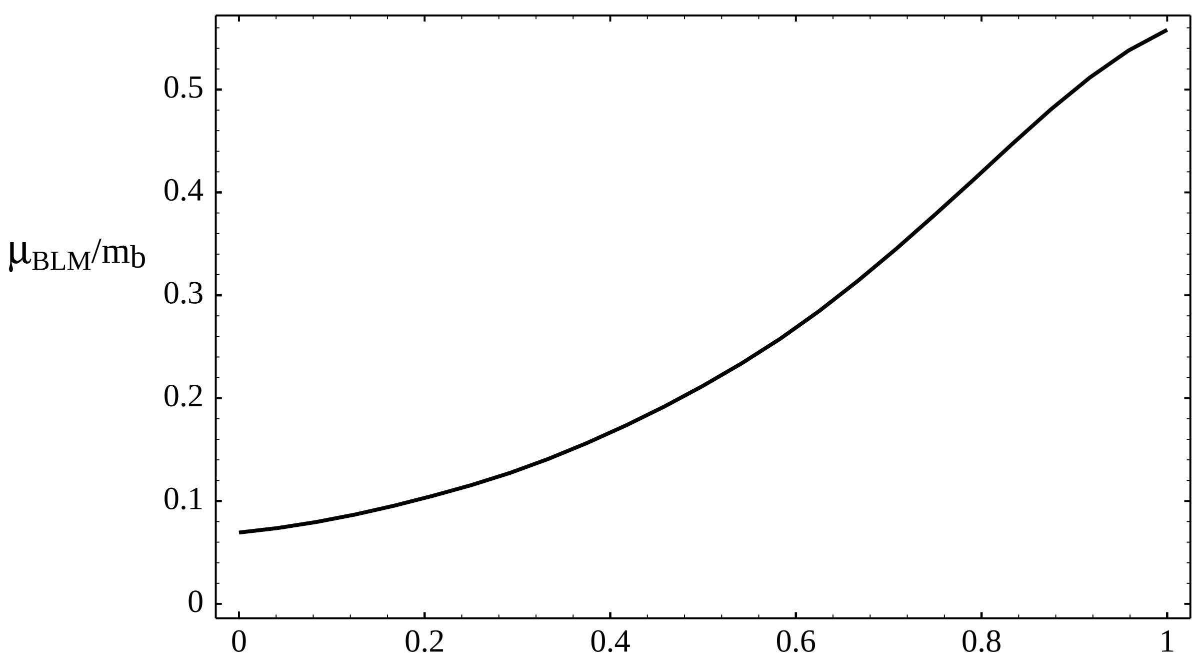
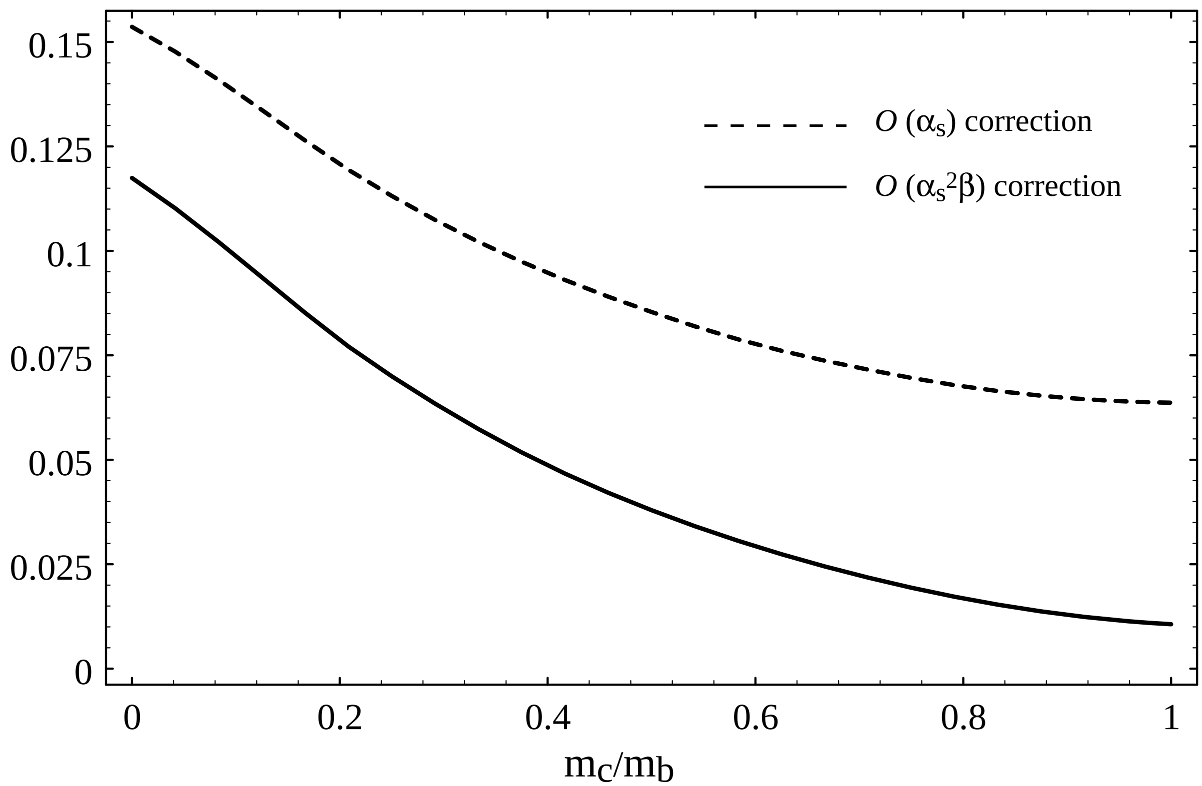


Figure 2



0.15
0.125
0.1
0.075
0.05
0.025
0
0
0.2
0.4
0.6
0.8
1
m_c/m_b
O (α_s) correction
O ($\alpha_s^2\beta$) correction

Figure 3

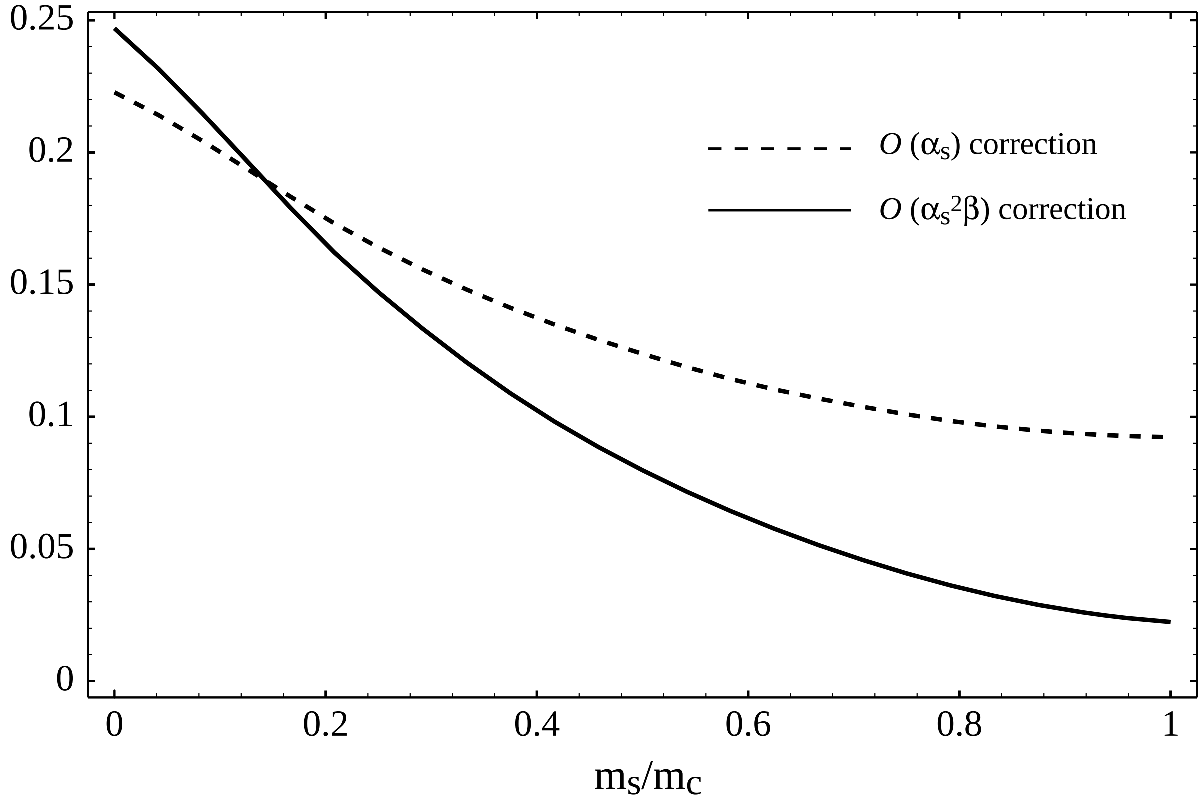

0.25
0.2
0.15
0.1
0.05
0
0
0.2
0.4
0.6
0.8
1
m_s/m_c
$O(\alpha_s)$ correction
$O(\alpha_s^2\beta)$ correction

Figure 4